ISRAEL JOURNAL OF MATHEMATICS 151 (2006), 357-380

THE EXCHANGE CONDITION FOR ASSOCIATION SCHEMES

BY

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ABSTRACT

The present article generalizes the group-theoretical exchange condition to the theory of association schemes. We prove that a large class of association schemes satisfying our exchange condition arises from groups as quotients over subgroups. The result provides an alternate proof of Tits' reduction theorem for buildings of spherical type.

Introduction

Let Γ be a group, and let Δ be a subgroup of Γ. For each element γ in Γ, we define γ^{Δ} to be the set of all pairs $(\beta\Delta, \beta\gamma\Delta)$ with $\beta \in \Gamma$. It is easy to see (and well-known) that $\{\gamma^{\Delta} | \ \gamma \in \Gamma\}$ is an association scheme (or a 'scheme', as we shall say briefly) with respect to $\{\gamma\Delta | \ \gamma \in \Gamma\}$.

Following [3] we call a scheme schurian if it arises from a pair of groups in the above-described way. It seems that a general scheme-theoretical condition which distinguishes schurian schemes within the class of all schemes is out of reach. It is for this reason that one might ask for specific conditions which force a scheme to be schurian.

In the present article, we focus on such a condition. In combination with other (general and natural) conditions, our condition turns out to be sufficient for a scheme to be schurian. We call our condition 'exchange condition', because it generalizes naturally the well-known group-theoretical exchange condition which distinguishes the Coxeter groups among the groups generated by involutions.

The generalization of the exchange condition from group theory to scheme theory is part of a major program in which basic concepts and results from group theory are generalized to scheme theory; cf., e.g., [4] and [5].

Received February 23, 2004

We wish to keep this note self-contained. Therefore, we shall start our considerations by recalling the definition of a scheme, and in order to do so we now fix a set X.

Let r be a subset of $X \times X$. We write r^* in order to denote the set of all pairs (y,z) such that $(z,y) \in r$. For each element x in X, we write xr for the set of all elements y in X which satisfy $(x,y) \in r$.

Let us fix a partition G of $X \times X$, and let us assume that $\emptyset \notin G$, that $1 \in G$, and that, for each element g in G, $g^* \in G$. (By 1 we mean the set of all pairs (x,x) where $x \in X$.) The set G is called a scheme with respect to X if, for any three elements d, e, and f in G, there exists a cardinal number a_{def} such that, for any two elements y in X and z in yf, $|yd \cap ze^*| = a_{def}$.

For the remainder of these introductory remarks, we shall now assume G to be a scheme with respect to X. We shall explain what it means for G to satisfy the above-mentioned exchange condition.

Let F be a non-empty subset of G. For each non-empty subset E of G, we write EF in order to denote the set of all elements g in G such that there exist elements e in E and f in F with $1 \leq a_{efg}$. If e stands for an element in G, we write eF instead of $\{e\}F$ and Fe instead of $F\{e\}$. The set F is called closed if, for each element f in F, $f^*F \subseteq F$. We define $\langle F \rangle$ to be the intersection of all closed subsets of G which contain F. We set $F^0 := \{1\}$. For each element n in $\mathbb{N} \setminus \{0\}$, we define $F^n := F^{n-1}F$.

An element g in $G \setminus \{1\}$ will be called an involution if $\{1, g\}$ is closed.

Let L be a set of involutions of G. It is easy to see that $\langle L \rangle$ is the union of the sets L^n where n is a non-negative integer; cf., e.g., [6; Theorem 1.4.1(i)]. In particular, for each element g in $\langle L \rangle$, there exists a smallest integer n such that $g \in L^n$; we denote this integer by $\ell_L(g)$.

If e and f stand for elements in G, we write ef instead of $e\{f\}$.

Let us now assume that $\langle L \rangle = G$. For each element e in G, we define $G_1(e)$ to be the set of all elements d in G such that there exists an element f in de with $\ell_L(f) = \ell_L(d) + \ell_L(e)$. We call G constrained over L if, for any two elements f in G and $e \in G_1(f)$, $1 = |ef|$.

Let us assume that G is constrained over L. We say that G satisfies the exchange condition with respect to L if, for any three elements h, k in L and g in $G_1(k)$, $h \in G_1(g)$ implies that $hg \subseteq gk \cup G_1(k)$. We call G a Coxeter scheme over L if G satisfies the exchange condition with respect to L.

The main results of this note are statements about Coxeter schemes. They deal with 'faithful' maps of Coxeter schemes. What is a faithful map?

Let W be a subset of X. A map χ from W to X is called faithful if, for any three elements y, z in W and g in G, $z \in yg$ implies that $z\chi \in y\chi g$.

Here is our first main result.

THEOREM A: *Let X be a set, let G be a scheme with respect to X, and let L be a set of involutions of G such that G is a Coxeter scheme over L. Let y be an element in X, g an element in G, and z an element in yg.*

Let $\mathcal{N}$ be a set of subsets of L. For each element N in $\mathcal{N}$, let N' be a subset of L such that, for $e \in g\langle N'\rangle \cap G_1(N')$, $\langle N'\rangle \subseteq \langle N\rangle^e$. Let V (respectively V') denote the union of the sets $\langle N\rangle$ (respectively $\langle N'\rangle$) with $N \in \mathcal{N}$.

Then, each faithful map from yV to X extends faithfully to $yV \cup zV'$.

A word about the specific notation used in Theorem A. Let F be a subset of G. We write $G_1(F)$ in order to denote the intersection of the sets $G_1(f)$ with $f \in F$. For each element g in G, we write F^g in order to denote the set of all elements e in G such that $ge \subseteq Fg$. For each element x in X, we define xF to be the union of the sets xf with $f \in F$.

For each subset F of G, we define $T(F)$ to be the set of all elements f in F such that $1 = |f^*f|$.

THEOREM B: *Let X be a set, let G be a scheme with respect to X, and let L be a set of involutions of G such that G is a Coxeter scheme over L. Let N be a subset of L such that $\langle N\rangle$ is finite and $T(N)$ is empty.*

Let V denote the union of the sets $\langle M\rangle$ with $M \subseteq N$ and $|M| \leq 2$, and let x be an element in X.

Then, each faithful map from xV to X extends faithfully to $x\langle N\rangle$.

Theorem B yields the following corollary.

COROLLARY: *Let G be a scheme, and let L be a set of involutions of G such that G is a Coxeter scheme over L. Assume that G is finite, that $3 \leq |L|$, and that $T(L)$ is empty. Then G is schurian.*

For the remainder of this note, the letter X will stand for a set and G for a scheme with respect to X.

1. Basic facts on schemes

We start with a few facts which we shall use occasionally without any reference.

First of all, it is clear that, for any three elements d, e, and f in G, the statements $f \in de$, $e \in d^*f$, and $d \in fe^*$ are pairwise equivalent. From this observation, one obtains easily that, for each closed subset H of G, $1 \in H$.

Moreover, for each closed subset H of G, $\{xH|\, x \in X\}$ is a partition of X and $\{gH|\, g \in G\}$ is a partition of G.

The following lemma is a special case of a well-known and general observation due to Richard Dedekind; cf. [2; Theorem VIII].

LEMMA 1.1: *Let H be a closed subset of G. Then, for any two subsets E and F of G with $F \subseteq H$, we have $H \cap EF = (H \cap E)F$.*

LEMMA 1.2: *Let H be a closed subset of G, and let e and f be elements in G.*

(i) *If $He = Hf$, $H^e = H^f$.*

(ii) *Let g be an element in G. Then, if e, $f \in H^g$, $ef \subseteq H^g$.*

Proof: (i) Let us assume that $He = Hf$. Then there exists an element h in H such that $f \in he$.

It is enough to show that $H^e \subseteq H^f$. In order to show this, we pick an element g in H^e, and we shall see that $g \in H^f$.

From $g \in H^e$ we obtain that $eg \subseteq He$. Thus, as $f \in he$, $fg \subseteq heg \subseteq hHe = He = Hf$. Thus, by definition, $g \in H^f$.

(ii) Let e and f be elements in H^g, let c be an element in ef. Since $e \in H^g$, $ge \subseteq Hg$. Since $f \in H^g$, $gf \subseteq Hg$. It follows that $gc \subseteq gef \subseteq Hgf \subseteq Hg$, and that means that $c \in H^g$. ∎

For the remainder of this note, the letter L will stand for a set of involutions of G.

LEMMA 1.3: *For each element f in $\langle L\rangle \setminus \{1\}$, there exist elements e in $\langle L\rangle$ and l in L such that $f \in el$ and $\ell_L(f) = \ell_L(e) + 1$.*

Proof: We set $n := \ell_L(f)$. Then, by definition, $f \in L^n$. On the other hand, as $1 \neq f$, $1 \leq n$. Thus, there exist elements e in L^{n-1} and l in L such that $f \in el$. From $e \in L^{n-1}$ we obtain that $\ell_L(e) \leq n-1$. From $n = \ell_L(f)$ and $f \in el$ we obtain that $n \leq \ell_L(e) + 1$. ∎

Let d, e, and f be elements in $\langle L\rangle$ such that $f \in de$. It is obvious that $\ell_L(f) \leq \ell_L(d) + \ell_L(e)$. In the following lemma, we focus on the the case where $\ell_L(f) = \ell_L(d) + \ell_L(e)$.

LEMMA 1.4: *Let d, e, and f be elements in $\langle L\rangle$ satisfying $f \in de$ and $\ell_L(f) = \ell_L(d) + \ell_L(e)$. Let b and c be elements in $\langle L\rangle$ satisfying $e \in bc$ and $\ell_L(e) = \ell_L(b) + \ell_L(c)$.*

Then, there exists an element g in db such that $f \in gc$, $\ell_L(g) = \ell_L(d) + \ell_L(b)$, and $\ell_L(f) = \ell_L(g) + \ell_L(c)$.

Proof: Since $f \in de$ and $e \in bc$, $f \in dbc$. Thus, there exists an element g in db such that $f \in gc$.

Since $g \in db$, $\ell_L(g) \leq \ell_L(d)+\ell_L(b)$. Since $f \in gc$, $\ell_L(f) \leq \ell_L(g)+\ell_L(c)$. Thus, as we are assuming that $\ell_L(e) = \ell_L(b) + \ell_L(c)$ and that $\ell_L(f) = \ell_L(d) + \ell_L(e)$,

$$\ell_L(f) \leq \ell_L(g) + \ell_L(c) \leq \ell_L(d) + \ell_L(b) + \ell_L(c) = \ell_L(f).$$

It follows that $\ell_L(g) = \ell_L(d) + \ell_L(b)$ and $\ell_L(f) = \ell_L(g) + \ell_L(c)$. ∎

For the remainder of this note, we shall assume that $\langle L \rangle = G$. Instead of ℓ_L we shall write ℓ.

Let e be an element in G. We define $G_{-1}(e)$ to be the set of all elements f in G such that there exists an element d in G with $f \in de$ and $\ell(f) = \ell(d) + \ell(e)$. (Recall that $G_1(e)$ stands for the set of all elements d in G such that there exists an element f in de with $\ell(f) = \ell(d) + \ell(e)$.)

Lemma 1.5: *For any two elements e and f in G, we have the following.*

(i) *If $f \in G_{-1}(e)$, $G_{-1}(f) \subseteq G_{-1}(e)$.*
(ii) *If $\emptyset \neq G_{-1}(e) \cap G_1(f)$, $e \in G_1(f)$.*
(iii) *If $e \in G_1(f)$, $f^* \in G_1(e^*)$.*
(iv) *If $f \in G_{-1}(e)$, $G_1(f^*) \subseteq G_1(e^*)$.*

Proof: (i) Let us assume that $f \in G_{-1}(e)$, and let us pick an element g in $G_{-1}(f)$. We shall show that $g \in G_{-1}(e)$.

Since $g \in G_{-1}(f)$, there exists an element d in G such that $g \in df$ and $\ell(g) = \ell(d) + \ell(f)$. On the other hand, we are assuming that $f \in G_{-1}(e)$. Thus, there exists an element c in G such that $f \in ce$ and $\ell(f) = \ell(c) + \ell(e)$. Now, by Lemma 1.4, there exists an element b in dc such that $g \in be$, $\ell(b) = \ell(d) + \ell(c)$, and $\ell(g) = \ell(b) + \ell(e)$. From $g \in be$ and $\ell(g) = \ell(b) + \ell(e)$ we now obtain that $g \in G_{-1}(e)$.

(ii) is another formal consequence of Lemma 1.4.

(iii) follows from the fact that, for each element g in G, $\ell(g^*) = \ell(g)$.

(iv) Let us assume that $f \in G_{-1}(e)$, and let us pick an element g in $G_1(f^*)$. We shall show that $g \in G_1(e^*)$.

Since $g \in G_1(f^*)$, $f \in G_1(g^*)$; cf. (iii). Thus, as we are assuming that $f \in G_{-1}(e)$, $e \in G_1(g^*)$; cf. (ii). Thus, by (iii), $g \in G_1(e^*)$. ∎

For the last two results of this section, we shall assume that, for any three elements h, k in L and g in $G_1(k)$, $h \in G_1(g)$ implies that $hg \subseteq G_{-1}(k) \cup G_1(k)$.

LEMMA 1.6: *For each element l in L, we have $G_{-1}(l) \cup G_1(l) = G$.*

Proof: Assume the claim to be false. Then $G \setminus (G_{-1}(l) \cup G_1(l))$ is not empty. Among the elements in $G \setminus (G_{-1}(l) \cup G_1(l))$ we choose g such that $\ell(g)$ is as small as possible.

Since $1 \in G_1(l)$ and $g \notin G_1(l)$, $1 \neq g$. Thus, by Lemma 1.3, there exist elements h in L and f in G such that $g \in hf$ and $\ell(g) = 1 + \ell(f)$. Since $\ell(g) = 1 + \ell(f)$, the (minimal) choice of g forces $f \in G_{-1}(l) \cup G_1(l)$.

Since $g \in hf$ and $\ell(g) = 1 + \ell(f)$, $g \in G_{-1}(f)$. Thus, as $g \notin G_{-1}(l)$, $f \notin G_{-1}(l)$; cf. Lemma 1.5(i). Thus, as $f \in G_{-1}(l) \cup G_1(l)$, $f \in G_1(l)$. On the other hand, as $g \in hf$ and $\ell(g) = 1 + \ell(f)$, $h \in G_1(f)$. Thus, by hypothesis, $hf \subseteq G_{-1}(l) \cup G_1(l)$. Thus, as $g \in hf$, $g \subseteq G_{-1}(l) \cup G_1(l)$, contradiction. ∎

LEMMA 1.7: *For each subset N of L, we have $G_1(N)\langle N\rangle = G$.*

Proof: Let us assume that $G_1(N)\langle N\rangle \neq G$. Then $G \setminus G_1(N)\langle N\rangle$ is not empty. Among the elements in $G \setminus G_1(N)\langle N\rangle$ we choose g such that $\ell(g)$ is as small as possible.

Since $g \notin G_1(N)\langle N\rangle$, $g \notin G_1(N)$. Thus, there exists an element l in N such that $g \notin G_1(l)$. Thus, by Lemma 1.6, $g \in G_{-1}(l)$. This means that there exists an element f in G such that $g \in fl$ and $\ell(g) = \ell(f) + 1$.

Since $\ell(g) = \ell(f) + 1$, the (minimal) choice of g yields $f \in G_1(N)\langle N\rangle$. Thus, as $g \in fl$ and $l \in N$, $g \in G_1(N)\langle N\rangle$, contradiction. ∎

2. Basic facts on constrained schemes

In this section, we assume G to be constrained over L.

For each non-empty subset F of G, we define $\ell(F)$ to be the set of all elements $\ell(f)$ with $f \in F$.

LEMMA 2.1: *For any two elements e and f in G, there exists at most one element d in G such that $f \in de$ and $\ell(f) = \ell(d) + \ell(e)$.*

Proof: Let us fix an element f in G. We shall denote by E the set of all elements e in G such that there exist elements d and d' in G with $f \in de$, $f \in d'e$, $\ell(f) = \ell(d) + \ell(e)$, $\ell(d') = \ell(d)$, and $d' \neq d$. By way of contradiction, we assume that $\emptyset \neq E$. We pick an element e in E which satisfies $\min \ell(E) = \ell(e)$.

Since $e \in E$, $1 \neq e$. Thus, by Lemma 1.3, there exist elements l in L and c in G such that $e \in lc$ and $\ell(e) = 1+\ell(c)$. Thus, as $f \in de$ and $\ell(f) = \ell(d)+\ell(e)$, there exists an element b in dl such that $f \in bc$, $\ell(b) = \ell(d)+1$, and $\ell(f) = \ell(b)+\ell(c)$; cf. Lemma 1.4.

Similarly, we find an element b' in $d'l$ such that $f \in b'c$, $\ell(b') = \ell(d')+1$, and $\ell(f) = \ell(b') + \ell(c)$.

Since $\ell(e) = 1 + \ell(c)$ and $\min \ell(E) = \ell(e)$, $c \notin E$. Thus, as $f \in bc$, $f \in b'c$, $\ell(f) = \ell(b) + \ell(c)$, and $\ell(f) = \ell(b') + \ell(c)$, $b' = b$.

We are assuming that G is constrained over L. Thus, as $b \in dl$ and $\ell(b) = \ell(d) + 1$, we have $\{b\} = dl$. Similarly, $\{b'\} = d'l$. Thus, as $b' = b$, $d'l = dl$. It follows that $d' \in \{d, b\}$. Thus, as $b' = b$ and $\ell(b') = \ell(d') + 1$, $d' = d$. This contradiction finishes the proof of the lemma. ∎

Lemma 2.2: *For any three elements d, e, and f in G such that $f \in de$ and $\ell(f) = \ell(d) + \ell(e)$, we have $a_{def} = 1$.*

Proof: Let us denote by F the set of the elements f in G such that there exist elements d and e in G with $f \in de$, $\ell(f) = \ell(d) + \ell(e)$, and $1 \neq a_{def}$. By way of contradiction, we assume that $\emptyset \neq F$. We pick an element f in F which satisfies $\min \ell(F) = \ell(f)$.

Since $f \in F$, there exist elements d and e in G such that $f \in de$, $\ell(f) = \ell(d) + \ell(e)$, and $1 \neq a_{def}$. Since $f \in de$ and $1 \neq a_{def}$, we have $2 \leq a_{def}$. In particular, $1 \neq d$ and $1 \neq e$.

Since $1 \neq e$, there exist elements c in G and l in L such that $e \in cl$, and $\ell(e) = \ell(c) + 1$; cf. Lemma 1.3.

Since $f \in de$, $\ell(f) = \ell(d) + \ell(e)$, $e \in cl$, and $\ell(e) = \ell(c) + 1$, there exists an element b in dc such that $f \in bl$, $\ell(b) = \ell(d) + \ell(c)$, and $\ell(f) = \ell(b) + 1$; cf. Lemma 1.4.

We are assuming that G is constrained over L. Thus, as $f \in bl$ and $\ell(f) = \ell(b) + 1$, $\{f\} = bl$.

It is easy to see (and well-known) that

$$\sum_{g \in G} a_{dcg} a_{glf} = \sum_{g \in G} a_{dgf} a_{clg};$$

cf. [1], [3], or [6; Lemma 1.1.3(i)].

Since $b \in dc$ and $\ell(b) = \ell(d) + \ell(c)$, $\{b\} = dc$. Thus, the left hand side of the above equation is equal to $a_{dcb} a_{blf}$.

Since $e \in cl$ and $\ell(e) = \ell(c) + 1$, $\{e\} = cl$. Thus, the right hand side of the above equation is equal to $a_{def}a_{cle}$.

The choice of f forces $a_{dcb} = 1$ and $a_{cle} = 1$. (Recall that $1 \neq d$. Therefore, $\ell(e) \leq \ell(f) - 1$.) Thus, $a_{blf} = a_{def}$. Thus, as $2 \leq a_{def}$, $2 \leq a_{blf}$. It follows that $b \in bl$. Since $\{f\} = bl$, this yields $f = b$, contrary to $\ell(f) = \ell(b) + 1$. ∎

LEMMA 2.3: *Let f be an element in G, and let e be an element in $G_1(f)$. Let x be an element in X, y an element in xe, and z an element in yf. Let χ be a map from $\{x, y, z\}$ to X.*

Then, if $\chi|_{\{x,y\}}$ and $\chi|_{\{y,z\}}$ are faithful, χ is faithful, too.

Proof: Let us denote by g the uniquely determined element in G which satisfies $z \in xg$. We have to prove that $z\chi \in x\chi g$.

Since $z \in yf$ and $y \in xe$, $z \in xef$. Thus, as $z \in xg$, $g \in ef$. Thus, as we are assuming that $e \in G_1(f)$, $\{g\} = ef$.

Assume that $\chi|_{\{x,y\}}$ is faithful. Then, as $y \in xe$, $y\chi \in x\chi e$. Assume that $\chi|_{\{y,z\}}$ is faithful. Then, as $z \in yf$, $z\chi \in y\chi f$. From $z\chi \in y\chi f$ and $y\chi \in x\chi e$ we obtain that $z\chi \in x\chi ef$. Thus, as $\{g\} = ef$, $z\chi \in x\chi g$. ∎

Let y and z be elements in X, and let n be the smallest element in $\mathbb{N}$ with $z \in yL^n$. We shall denote by $S(y, z)$ the union of the sets $yL^i \cap zL^j$ which satisfy $i + j = n$.

LEMMA 2.4: *Let y and z be elements in X, let v be an element in $S(y, z)$, and let w be an element in $S(v, z)$. Let χ be a map from $\{v, w, y, z\}$ to X.*

Then, if $\chi|_{\{y,v,z\}}$ and $\chi|_{\{y,w,z\}}$ are faithful, χ is faithful, too.

Proof: Let us denote by d the uniquely determined element in G which satisfies $v \in yd$, by b the one which satisfies $w \in vb$. Then, we have $w \in ydb$. Thus, there exists an element g in db such that $w \in yg$.

Since $w \in yg$, $w\chi \in y\chi g$. (We are assuming that $\chi|_{\{y,w,z\}}$ is faithful.) Thus, as $g \in db$, $w\chi \in y\chi db$. Thus, there exists an element x in $y\chi d$ such that $w\chi \in xb$.

Let us denote by c the uniquely determined element in G which satisfies $z \in wc$. Then, as $w \in vb$, $z \in vbc$. Thus, there exists an element e in bc such that $z \in ve$. From $e \in bc$ and $w \in S(v, z)$ we obtain that $\ell(e) = \ell(b) + \ell(c)$. Thus, $\{e\} = bc$.

Since $z \in wc$, $z\chi \in w\chi c$. (Again, we use the hypothesis that $\chi|_{\{y,w,z\}}$ is faithful.) Thus, as $w\chi \in xb$, $z\chi \in xbc$. Thus, as $\{e\} = bc$, $z\chi \in xe$.

From $z \in ve$ and $v \in yd$ we obtain that $z \in yde$. Thus, there exists an element f in de such that $z \in yf$. From $f \in de$ and $v \in S(y,z)$ we obtain that $\ell(f) = \ell(d) + \ell(e)$. Thus, by Lemma 2.2, $a_{def} = 1$.

Since $z \in yf$, $z\chi \in y\chi f$. Since $v \in yd \cap ze^*$, $v\chi \in y\chi d \cap z\chi e^*$. (This time, we use that $\chi|_{\{y,v,z\}}$ is assumed to be faithful.) On the other hand, we also have $x \in y\chi d \cap z\chi e^*$. Thus, as $a_{def} = 1$, $v\chi = x$. Thus, as $w\chi \in xb$, $w\chi \in v\chi b$. ∎

From now on, we assume G to be a Coxeter scheme over L. In particular, we may apply all results of the first two sections.

3. Basic facts on Coxeter schemes

In this section, we collect general results on Coxeter schemes. Our first result is a formal generalization of the exchange condition.

LEMMA 3.1: *Let h be an element in L, and let c, d be elements in G such that $d \in hc$ and $\ell(d) = 1 + \ell(c)$. Let k be an element in L, and let e, f be elements in G such that $f \in ek$ and $\ell(f) = \ell(e) + 1$.*

Then, if $d \in G_1(e)$ and $c \in G_1(f)$, we have $de = cf$ or $d \in G_1(f)$.

Proof: Assume that $d \in G_1(e)$. Then, there exists an element g in de such that $\ell(g) = \ell(d) + \ell(e)$. Since $d \in hc$ and $\ell(d) = 1 + \ell(c)$, there exists an element b in ce such that $g \in hb$, $\ell(b) = \ell(c) + \ell(e)$, and $\ell(g) = 1 + \ell(b)$; cf. Lemma 1.4. From $b \in ce$ and $\ell(b) = \ell(c) + \ell(e)$ we obtain that $\{b\} = ce$. From $g \in hb$ and $\ell(g) = 1 + \ell(b)$ we obtain that $h \in G_1(b)$.

Similarly, using $\{b\} = ce$, we conclude from $c \in G_1(f)$ that $b \in G_1(k)$. Thus, as G is assumed to be a Coxeter scheme over L, we now have $hb = bk$ or $hb \subseteq G_1(k)$.

Since $\{d\} = hc$, $\{b\} = ce$, and $\{f\} = ek$, the first case yields $de = cf$.

Since $g \in hb$, the second case yields $g \in G_1(k)$. Thus, by definition, there exists an element a in gk such that $\ell(a) = \ell(g) + 1$. Since $a \in gk$ and $gk = hbk = hcek = df$, $a \in df$. Since $\ell(a) = \ell(g) + 1$ and $\ell(g) + 1 = 1 + \ell(b) + 1 = 1 + \ell(c) + \ell(e) + 1 = \ell(d) + \ell(f)$, $\ell(a) = \ell(d) + \ell(f)$. Thus, $d \in G_1(f)$. ∎

LEMMA 3.2: *Let l be an element in L, let e be an element in $G_1(l)$, and let f stand for the element in el. Then $G_{-1}(e) \cap G_{-1}(l) \subseteq G_{-1}(f)$.*

Proof: Let us denote by E the set of all elements g in $G_{-1}(e) \cap G_{-1}(l)$ with $g \notin G_{-1}(f)$. By way of contradiction, we assume that $\emptyset \neq E$. We pick an element g in E such that $\min \ell(E) = \ell(g)$.

Since $g \in G_{-1}(e)$, there exists an element c in G such that $g \in ce$ and $\ell(g) = \ell(c) + \ell(e)$. Since $e \in G_1(l)$ and $g \in G_{-1}(l)$, $e \neq g$. Thus, as $g \in ce$, $1 \neq c$. Thus, by Lemma 1.3, there exist elements h in L and b in G such that $c \in hb$ and $\ell(c) = 1 + \ell(b)$. Thus, by Lemma 1.4, there exists an element g' in be such that $g \in hg'$, $\ell(g') = \ell(b) + \ell(e)$, and $\ell(g) = 1 + \ell(g')$.

Suppose that $g' \in G_{-1}(e) \cap G_{-1}(l)$. Then, as $g' \notin E$, $g' \in G_{-1}(f)$. On the other hand, as $g \in hg'$ and $\ell(g) = 1 + \ell(g')$, $g \in G_{-1}(g')$. Thus, by Lemma 1.5(i), $g \in G_{-1}(f)$, contrary to $g \in E$.

This contradiction forces $g' \notin G_{-1}(e) \cap G_{-1}(l)$. On the other hand, as $g' \in be$ and $\ell(g') = \ell(b) + \ell(e)$, $g' \in G_{-1}(e)$. Thus, $g' \notin G_{-1}(l)$. Thus, by Lemma 1.6, $g' \in G_1(l)$. Thus, by our latest hypothesis, $hg' \subseteq g'l \cup G_1(l)$. Thus, as $g \in hg'$, $g \in g'l \cup G_1(l)$. Thus, as $g \in G_{-1}(l)$, $g \in g'l \subseteq bel$. Thus, there exists an element f in el such that $g \in bf$. It follows that

$$\ell(g) \leq \ell(b) + \ell(f) \leq \ell(b) + \ell(e) + 1 = \ell(g') + 1 = \ell(g).$$

This forces $\ell(g) = \ell(b) + \ell(f)$. Thus, as $g \in bf$, $g \in G_{-1}(f)$, contrary to $g \in E$. ∎

For the remainder of this section, the letter N stands for a subset of L.

LEMMA 3.3: *For each element g in $\langle N \rangle$, $\ell(g) = \ell_N(g)$.*

Proof: Assume the claim to be false. Among the elements in $\langle N \rangle$ which do not satisfy the equation in question we choose g in such a way that $\ell_N(g)$ is as small as possible.

Since $\ell(g) \neq \ell_N(g)$, $1 \neq g$. Thus, by Lemma 1.3, there exist elements h in N and f in $\langle N \rangle$ such that $g \in hf$ and $\ell_N(g) = 1 + \ell_N(f)$.

Since $\ell(g) \neq \ell_N(g)$, $g \notin N$. Thus, as $g \in hf$ and $h \in N$, $1 \neq f$. Thus, by Lemma 1.3, there exist elements e in $\langle N \rangle$ and k in N such that $f \in ek$ and $\ell_N(f) = \ell_N(e) + 1$. Now, by Lemma 1.4, there exists an element d in he such that $g \in dk$, $\ell_N(d) = 1 + \ell_N(e)$, and $\ell_N(g) = \ell_N(d) + 1$.

Since $\ell_N(g) = \ell_N(d) + 1$, the (minimal) choice of g yields $\ell(d) = \ell_N(d)$. Similarly, as $\ell_N(g) = 1 + \ell_N(f)$ and $\ell_N(f) = \ell_N(e) + 1$, $\ell(e) = \ell_N(e)$. Thus, as $d \in he$ and $\ell_N(d) = 1 + \ell_N(e)$, $h \in G_1(e)$.

Similarly, one obtains that $e \in G_1(k)$. Thus, as G is assumed to be a Coxeter scheme over L, we obtain that $he = ek$ or that $he \subseteq G_1(k)$.

Since $g \in hek$, the first of these two cases yields $g \in ekk = \{e\} \cup ek$, contrary to $\ell_N(g) = \ell_N(e) + 2$. Since $d \in he$, the second case yields $d \in G_1(k)$. Thus, as $g \in dk$, $\ell(g) = \ell(d) + 1$. (Here we use the hypothesis that G is constrained over L.) Thus, as $\ell(d) = \ell_N(d)$ and $\ell_N(g) = \ell_N(d) + 1$, $\ell(g) = \ell_N(g)$. This contradiction finishes the proof of the lemma. ∎

If F stands for a subset of G, we shall write $G_{-1}(F)$ in order to denote the intersection of the sets $G_{-1}(f)$ with $f \in F$.

LEMMA 3.4: *We have* $G_{-1}(\langle N\rangle) = G_{-1}(N)$.

Proof: Let us assume that $G_{-1}(\langle N\rangle) \neq G_{-1}(N)$. Then, as $G_{-1}(\langle N\rangle) \subseteq G_{-1}(N)$, $G_{-1}(N) \not\subseteq G_{-1}(\langle N\rangle)$. Thus, there exists an element g in $G_{-1}(N)$ such that $g \notin G_{-1}(\langle N\rangle)$.

Since $g \notin G_{-1}(\langle N\rangle)$, there exists an element f in $\langle N\rangle$ such that $g \notin G_{-1}(f)$. Among the elements f in $\langle N\rangle$ satisfying $g \notin G_{-1}(f)$ we pick f such that $\ell(f)$ is as small as possible.

Since $g \notin G_{-1}(f)$, $1 \neq f$. Thus, Lemma 1.2 gives us elements e in $\langle N\rangle$ and k in N such that $f \in ek$ and $\ell(f) = \ell(e) + 1$. Now, the minimal choice of f forces $g \in G_{-1}(e)$. Thus, as $g \in G_{-1}(N) \subseteq G_{-1}(l)$, $g \in G_{-1}(f)$; cf. Lemma 3.2. This contradiction finishes our proof. ∎

LEMMA 3.5: *We have* $G_1(\langle N\rangle) = G_1(N)$.

Proof: Let us assume that $G_1(\langle N\rangle) \neq G_1(N)$. Then, as $G_1(\langle N\rangle) \subseteq G_1(N)$, $G_1(N) \not\subseteq G_1(\langle N\rangle)$. Among the elements in $G_1(N) \setminus G_1(\langle N\rangle)$ we choose d such that $\ell(d)$ is as small as possible.

Since $1 \in G_1(\langle N\rangle)$ and $d \notin G_1(\langle N\rangle)$, $1 \neq d$. Thus, by Lemma 1.3, there exist elements h in L and c in G such $d \in hc$ and $\ell(d) = 1 + \ell(c)$.

Since $d \notin G_1(\langle N\rangle)$, there exists an element f in $\langle N\rangle$ such that $d \notin G_1(f)$. Among the elements f in $\langle N\rangle$ satisfying $d \notin G_1(f)$ we choose f in such a way that $\ell(f)$ is minimal. Since $d \notin G_1(f)$, $1 \neq f$. Thus, by Lemma 1.3 and Lemma 3.3, there exist elements e in $\langle N\rangle$ and k in N such that $f \in ek$ and $\ell(f) = \ell(e) + 1$.

Since $d \in hc$ and $\ell(d) = 1+\ell(c)$, $d \in G_{-1}(c)$. Thus, as $d \in G_1(N)$, $c \in G_1(N)$; cf. Lemma 1.5(ii). Thus, as $\ell(d) = 1 + \ell(c)$, the (minimal) choice of d yields $c \in G_1(\langle N\rangle)$.

Since $\ell(f) = \ell(e) + 1$ and $e \in \langle N\rangle$, the (minimal) choice of f yields $d \in G_1(e)$. Since $c \in G_1(\langle N\rangle)$ and $f \in \langle N\rangle$, $c \in G_1(f)$. Thus, by Lemma 3.1, $de = cf$ or $d \in G_1(f)$. Thus, by the choice of f, $de = cf$. Thus, as e, $f \in \langle N\rangle$, $d \in c\langle N\rangle$. Thus, there exists an element g in $\langle N\rangle$ such that $d \in cg$.

Since $c \in G_1(\langle N\rangle)$ and $g \in \langle N\rangle$, $c \in G_1(g)$. Thus, as $d \in cg$, $\ell(d) = \ell(c)+\ell(g)$. Since $\ell(d) = 1+\ell(c)$, this means that $\ell(g) = 1$. Thus, by Lemma 3.3, $g \in N$. On the other hand, as $d \in cg$ and $\ell(d) = \ell(c) + \ell(g)$, $d \in G_{-1}(g)$. Thus, $d \notin G_1(N)$, contrary to our choice of d. ∎

LEMMA 3.6: *We have* $\langle L \setminus N\rangle \subseteq G_1(\langle N\rangle)$.

Proof: Let l be an element in $L \setminus N$. Then as G is assumed to be a Coxeter scheme over L, $l \in G_1(N)$. Thus, by Lemma 3.5, $l \in G_1(\langle N\rangle)$. Thus, by Lemma 1.5(iii), $\langle N\rangle \subseteq G_1(l)$.

Since l has been chosen arbitrarily in $L \setminus N$, we have shown that $\langle N\rangle \subseteq G_1(L \setminus N)$. Thus, by Lemma 3.5, $\langle N\rangle \subseteq G_1(\langle L \setminus N\rangle)$. Thus, by Lemma 1.5(iii), $\langle L \setminus N\rangle \subseteq G_1(\langle N\rangle)$. ∎

LEMMA 3.7: *For each subset M of L,* $(\langle M\rangle \cap G_1(N))\langle M \cap N\rangle = \langle M\rangle$.

Proof: From Lemma 3.6 we know that $\langle M\rangle \subseteq G_1(N \setminus M)$. Thus,

$$\langle M\rangle \cap G_1(M \cap N) = \langle M\rangle \cap G_1(N).$$

On the other hand, Lemma 1.7 says that $\langle M\rangle \cap G_1(M \cap N)\langle M \cap N\rangle = \langle M\rangle$, and according to Lemma 1.1, the left hand side of this equation is equal to $(\langle M\rangle \cap G_1(M \cap N))\langle M \cap N\rangle$. ∎

LEMMA 3.8: *Let e and f be elements in G, and let l be an element in L such that $l \in G_1(e)$ and $l \in G_1(f)$. Then, if $le \subseteq lf\langle N\rangle$, $e \in f\langle N\rangle$.*

Proof: Let us assume, by way of contradiction, that $le \subseteq lf\langle N\rangle$ and $e \notin f\langle N\rangle$. Since $le \subseteq lf\langle N\rangle$, we have $e \in f\langle N\rangle$ or $e \in lf\langle N\rangle$. Thus, as $e \notin f\langle N\rangle$, $e \in lf\langle N\rangle$.

By Lemma 1.7, there exists an element d in $G_1(N)$ such that $f \in d\langle N\rangle$. By Lemma 3.5, $f^* \in G_{-1}(d^*)$. Thus, as $l \in G_1(f)$, $l \in G_1(d)$; cf. Lemma 1.5(iv). Thus, as G is assumed to be a Coxeter scheme over L, we have $ld \subseteq dN$ or $ld \subseteq G_1(N)$. Since $e \in lf\langle N\rangle$, $f\langle N\rangle = d\langle N\rangle$, and $e \notin f\langle N\rangle$, we cannot have $ld \subseteq dN$. Thus, $ld \subseteq G_1(N)$.

Similarly, we obtain an element c in $G_1(N)$ such that $e \in c\langle N\rangle$, $l \in G_1(c)$, and $lc \subseteq G_1(N)$. Using Lemma 3.5 once again we now obtain that $lc = ld$. (Note that $lc\langle N\rangle = ld\langle N\rangle$.) Thus, by Lemma 2.1, $c = d$. Thus, as $e \in c\langle N\rangle$ and $f \in d\langle N\rangle$, $e \in f\langle N\rangle$, contradiction. ∎

LEMMA 3.9: *Let l be an element in L, and let e be an element in $G_1(N)$. Assume there exists an element g in $e\langle N\rangle$ such that $l \in G_1(g) \setminus \langle N\rangle^{g^*}$. Then $le \subseteq G_1(N)$.*

Proof: From $g \in e\langle N\rangle$ we obtain that $\langle N\rangle g^* = \langle N\rangle e^*$. Thus, by Lemma 1.2(i), $\langle N\rangle^{g^*} = \langle N\rangle^{e^*}$. Thus, as we are assuming that $l \notin \langle N\rangle^{g^*}$, $l \notin \langle N\rangle^{e^*}$. That means that $e^*l \not\subseteq \langle N\rangle e^*$, so that we have $le \not\subseteq e\langle N\rangle$.

On the other hand, we are assuming that $l \in G_1(g)$. Thus, by Lemma 1.5(ii), (iii), $l \in G_1(e)$. Thus, as G is assumed to be a Coxeter scheme over L, we have $le \subseteq G_1(N)$. ∎

LEMMA 3.10: *For any two elements y and z in X, $|yG_1(N) \cap z\langle N\rangle| = 1$.*

Proof: From Lemma 1.7 we know that $yG_1(N) \cap z\langle N\rangle$ is not empty. In order to show that $yG_1(N) \cap z\langle N\rangle$ has exactly one element, we now pick elements v and w in $yG_1(N) \cap z\langle N\rangle$, and we shall see that $v = w$.

Since $v \in yG_1(N)$, there exists an element e in $G_1(N)$ such that $v \in ye$. Since $v, w \in z\langle N\rangle$, $w \in v\langle N\rangle$. Thus, there exists an element c in $\langle N\rangle$ such that $w \in vc$. Together, this yields $w \in yec$. Thus, there exists an element f in ec such that $w \in yf$. Since $e \in G_1(N)$, $c \in \langle N\rangle$, and $f \in ec$, $\ell(f) = \ell(e) + \ell(c)$; cf. Lemma 3.5.

From $w \in yG_1(N)$ we similarly obtain that $\ell(e) = \ell(f)+\ell(c^*)$. Thus, $0 = \ell(c)$. It follows that $1 = c$. Thus, as $w \in vc$, $v = w$. ∎

If g stands for an element in G, we shall write $\langle g\rangle$ instead of $\langle\{g\}\rangle$.

LEMMA 3.11: *Let l be an element in L, let x, y, z be elements in X such that $x\langle l\rangle = y\langle l\rangle = z\langle l\rangle$ and $y \neq z$. Then, for each subset M of L, $y\langle M\rangle \cap z\langle N\rangle \subseteq x(\langle M\rangle \cup \langle N\rangle)$.*

Proof: Let w be an element in $y\langle M\rangle \cap z\langle N\rangle$. Since $w \in z\langle N\rangle$, $z \in w\langle N\rangle$. Thus, as $w \in y\langle M\rangle$, $z \in y\langle M\rangle\langle N\rangle$. Thus, as $z \in yl$, $l \in \langle M\rangle\langle N\rangle \subseteq \langle M \cup N\rangle$. It follows that $l \in M \cup N$; cf. Lemma 3.3.

Since $w \in y\langle M\rangle$ and $y \in x\langle l\rangle$, $w \in x\langle l\rangle\langle M\rangle$. Similarly, $w \in x\langle l\rangle\langle N\rangle$. Thus, as $l \in M \cup N$, we must have $w \in x\langle M\rangle$ or $w \in x\langle N\rangle$. Thus, $w \in x(\langle M\rangle \cup \langle N\rangle)$. ∎

4. Faithful maps and Coxeter schemes

In this section, the letter N stands for a subset of L.

LEMMA 4.1: *Let y be an element in X, let z be an element in $yG_1(N)$, and let χ be a map from $\{y\} \cup z\langle N\rangle$ to X.*

Then, if $\chi|_{\{y,z\}}$ and $\chi|_{z\langle N\rangle}$ are faithful, χ is faithful, too.

Proof: Let x be an element in $z\langle N\rangle$. We shall be done if we succeed in showing that $\chi|_{\{y,x\}}$ is faithful.

Since $x \in z\langle N\rangle$, there exists an element f in $\langle N\rangle$ such that $x \in zf$. Since we are assuming that $z \in yG_1(N)$, we find an element e in $G_1(N)$ such that $z \in ye$. Since $e \in G_1(N)$, $e \in G_1(\langle N\rangle)$; cf. Lemma 3.5. Thus, as $f \in \langle N\rangle$, $e \in G_1(f)$. Thus, as $\chi|_{\{y,z\}}$ and $\chi|_{\{z,x\}}$ are faithful, $\chi|_{\{y,x\}}$ is faithful; cf. Lemma 2.3. ∎

LEMMA 4.2: *Let x be an element in X, let M be a subset of L, and let χ be a map from $x(\langle M\rangle \cup \langle N\rangle)$ to X.*

Then, if $\chi|_{x\langle M\rangle}$ and $\chi|_{x\langle N\rangle}$ are faithful, χ is faithful, too.

Proof: Let y be an element in $x\langle M\rangle$. Then, $x \in y\langle M\rangle$. Thus, by Lemma 3.7, $x \in y(\langle M\rangle \cap G_1(N))\langle N\rangle$. Thus, there exists an element z in $y(\langle M\rangle \cap G_1(N))$ such that $x \in z\langle N\rangle$.

Since $z \in y\langle M\rangle$, $\chi|_{\{y,z\}}$ is faithful. Since $x \in z\langle N\rangle$, $\chi|_{z\langle N\rangle}$ is faithful. Thus, as $z \in yG_1(N)$, $\chi|_{\{y\}\cup x\langle N\rangle}$ must be faithful; cf. Lemma 4.1. ∎

LEMMA 4.3: *Let g be an element in G, and let l be an element in L with $l \in G_1(g) \setminus \langle N\rangle^{g^*}$. Let x be an element in X, y an element in xl, and z an element in yg. Finally, let χ be a map from $\{x,y\} \cup z\langle N\rangle$ to X.*

Then, if $\chi|_{\{x,y\}}$ and $\chi|_{\{y\}\cup z\langle N\rangle}$ are faithful, χ is faithful, too.

Proof: By Lemma 1.7, there exists an element e in $G_1(N)$ such that $g \in e\langle N\rangle$. Thus, as we are assuming that $l \in G_1(g) \setminus \langle N\rangle^{g^*}$, $le \subseteq G_1(N)$; cf. Lemma 3.9.

Since $z \in yg$ and $g \in e\langle N\rangle$, $z \in ye\langle N\rangle$. Thus, there exists an element w in ye such that $z \in w\langle N\rangle$. Since $w \in ye$ and $y \in xl$, $w \in xle$. Thus, as $le \subseteq G_1(N)$, $w \in xG_1(N)$.

By hypothesis, $\chi|_{\{x,y\}}$ and $\chi|_{\{y,w\}}$ are faithful. Moreover, as $l \in G_1(g)$, $l \in G_1(e)$; cf. Lemma 1.5(iv). Thus, as $y \in xl$ and $w \in ye$, $\chi|_{\{x,w\}}$ is faithful, too; cf. Lemma 2.3. On the other hand, as $z \in w\langle N\rangle$, $\chi|_{w\langle N\rangle}$ is faithful. Thus, as $w \in xG_1(N)$ and $z \in w\langle N\rangle$, $\chi|_{\{x\}\cup z\langle N\rangle}$ is faithful, too; cf. Lemma 4.1. ∎

LEMMA 4.4: *Let M be a subset of L, and let g be an element in $G_1(M)$ such that $\langle M\rangle \subseteq \langle N\rangle^g$. Let y be an element in X, let z be an element in $yg\langle M\rangle$, and let z' be an element in $z\langle M\rangle$.*

Then, for each faithful map χ from $y\langle N\rangle \cup \{z\}$ to X, there exists at most one faithful map χ' from $y\langle N\rangle \cup \{z'\}$ to X, such that $\chi'|_{y\langle N\rangle} = \chi|_{y\langle N\rangle}$ and $z'\chi' \in z\chi\langle M\rangle$.

Proof: We are assuming that $z \in yg\langle M\rangle$. Thus, $z' \in yg\langle M\rangle$. On the other hand, we are assuming that $\langle M\rangle \subseteq \langle N\rangle^g$, and that means that $g\langle M\rangle \subseteq \langle N\rangle g$. Thus, $z' \in y\langle N\rangle g$, so that we find an element y' in $y\langle N\rangle$ with $z' \in y'g$.

Let us now fix a faithful map χ' from $y\langle N\rangle \cup \{z'\}$ to X which satisfies $\chi'|_{y\langle N\rangle} = \chi|_{y\langle N\rangle}$ and $z'\chi' \in z\chi\langle M\rangle$.

Since $y' \in y\langle N\rangle$, χ' is defined on y'. Moreover, we have $z' \in y'g$. Thus, as χ' is assumed to be faithful, we have $z'\chi' \in y'\chi'g$. On the other hand, as we are assuming that $\chi'|_{y\langle N\rangle} = \chi|_{y\langle N\rangle}$, we have $y'\chi' = y'\chi$. Thus, $z'\chi' \in y'\chi g$. Thus, as $g \in G_1(M)$, $z'\chi' \in y'\chi G_1(M)$.

Thus, as $z'\chi' \in z\chi\langle M\rangle$, the claim follows from Lemma 3.10. ∎

5. Proof of Theorem A

It is the goal of this section to prove Theorem A. The main idea is the use of Corollary 5.3 in the proof of Lemma 5.5.

In this section, the letter $\mathcal{N}$ stands for a non-empty set of subsets of L.

LEMMA 5.1: *We have $\langle\bigcap_{N\in\mathcal{N}} N\rangle = \bigcap_{N\in\mathcal{N}}\langle N\rangle$.*

Proof: Let us denote by D the intersection of the elements in $\mathcal{N}$ and by H the intersection of the sets $\langle N\rangle$ where $N \in \mathcal{N}$. We have to show that $\langle D\rangle = H$.

By way of contradiction, we assume that $\langle D\rangle \neq H$. Then, as $\langle D\rangle \subseteq H$, $H \not\subseteq \langle D\rangle$. We pick an element g in $H\setminus\langle D\rangle$ which satisfies $\min \ell(H\setminus\langle D\rangle) = \ell(g)$.

Since $1 \in \langle D\rangle$ and $g \notin \langle D\rangle$, $1 \neq g$. Thus, by Lemma 1.3, there exist elements f in G and l in L such that $g \in fl$ and $\ell(g) = \ell(f)+1$. It follows that $g \in G_{-1}(l)$. Thus, as $g \in H$, $l \in D$; cf. Lemma 3.6. Thus, as $g \in fl$ and $g \in H$, $f \in H$. Thus, as $\ell(g) = \ell(f) + 1$ and $\min \ell(H \setminus \langle D\rangle) = \ell(g)$, $f \in \langle D\rangle$. Thus, as $g \in fl$ and $l \in D$, $g \in \langle D\rangle$, contradiction. ∎

LEMMA 5.2: *For each element g in G, we have $g\langle\bigcap_{N\in\mathcal{N}} N\rangle = \bigcap_{N\in\mathcal{N}} g\langle N\rangle$.*

Proof: Let us denote by E the set of all elements g in G which do not satisfy the equation in question. By way of contradiction, we assume that $\emptyset \neq E$. We pick an element g in E which satisfies $\min \ell(E) = \ell(g)$.

By Lemma 5.1, $1 \notin E$. Thus, as $g \in E$, $1 \neq g$. Thus, by Lemma 1.3, there exist elements l in L and f in G such that $g \in lf$ and $\ell(g) = 1 + \ell(f)$.

Let us denote by D the intersection of the elements in $\mathcal{N}$. By F we shall denote the intersection of the sets $g\langle N\rangle$ where $N \in \mathcal{N}$. Then, as $g \in E$, $g\langle D\rangle \neq F$. Thus, as $g\langle D\rangle \subseteq F$, $F \not\subseteq g\langle D\rangle$. Thus, we find an element d in F such that $d \notin g\langle D\rangle$.

Let us first assume that $d^* \in G_{-1}(l)$. Then, there exists an element c in G such that $d \in lc$ and $\ell(d) = 1 + \ell(c)$. Thus, $l \in G_1(c)$. Moreover, since $g \in lf$ and $\ell(g) = 1+\ell(f)$, $l \in G_1(f)$. On the other hand, for each element N in $\mathcal{N}$, we have $d \in g\langle N\rangle$. Thus, for each element N in $\mathcal{N}$, we have $c \in f\langle N\rangle$; cf. Lemma 3.8. Thus, as $\ell(g) = 1 + \ell(f)$ and $\min \ell(E) = \ell(g)$, $c \in f\langle D\rangle$. It follows that $d \in lc \subseteq lf\langle D\rangle = g\langle D\rangle$, contrary to the choice of d.

Let us now assume that $d^* \notin G_{-1}(l)$. Then, by Lemma 1.6, $d^* \in G_1(l)$. Thus, by Lemma 1.5(iii), $l \in G_1(d)$. Thus, there exists an element e in G such that $\{e\} = ld$. Since $d \in F$, we have that, for each element N in $\mathcal{N}$, $d \in g\langle N\rangle$. Thus, for each element N in $\mathcal{N}$, $f \in lg \subseteq ld\langle N\rangle = e\langle N\rangle$, and this is equivalent to $e \in f\langle N\rangle$. Thus, as $\ell(g) = 1 + \ell(f)$ and $\min \ell(E) = \ell(g)$, $e \in f\langle D\rangle$. Thus, $d \in le \subseteq lf\langle D\rangle = g\langle D\rangle$, contrary to the choice of d. ∎

COROLLARY 5.3: *For each element g in G, we have $\langle\bigcap_{N\in\mathcal{N}} N\rangle^g = \bigcap_{N\in\mathcal{N}}\langle N\rangle^g$.*

Proof: Let us denote by D the intersection of the elements in $\mathcal{N}$. Then, for each element N in $\mathcal{N}$, $\langle D\rangle^g \subseteq \langle N\rangle^g$.

Conversely, let e be an element in G such that, for each element N in $\mathcal{N}$, $e \in \langle N\rangle^g$. Then, for each element N in $\mathcal{N}$, $ge \subseteq \langle N\rangle g$. Thus, by Lemma 5.2, $ge \subseteq \langle D\rangle g$, and this means that $e \in \langle D\rangle^g$. ∎

For the remainder of this section, we denote by V the union of the sets $\langle N\rangle$ with $N \in \mathcal{N}$. We also fix two elements y and z in X.

LEMMA 5.4: *Let x be an element in $S(y, z)$. Then, each faithful map from $\{y\} \cup zV$ to X extends faithfully to $\{x, y\} \cup zV$.*

Proof: Let d (respectively e) stand for the uniquely determined element in G which satisfies $x \in yd$ (respectively $z \in xe$). Then, we have $z \in yde$. Thus, there exists an element f in de such that $z \in yf$.

Let χ be a faithful map from $\{y\} \cup zV$ to X. Then, as $z \in yf$, $z\chi \in y\chi f$. Thus, as $f \in de$, $z\chi \in y\chi de$. Therefore, there exists an element v in $y\chi d$ such that $z\chi \in ve$. We set $x\chi := v$. Then, $\chi|_{\{y,x,z\}}$ is faithful.

Now we pick an element N in $\mathcal{N}$. Then, by Lemma 1.7, $z \in xG_1(N)\langle N\rangle$. Thus, there exists an element w in $xG_1(N)$ such that $z \in w\langle N\rangle$. Thus, by Lemma 3.5, $w \in S(x,z)$. On the other hand, as $w \in z\langle N\rangle \subseteq zV$, $\chi|_{\{y,w,z\}}$ is faithful. Thus, as $x \in S(y,z)$ and $\chi|_{\{y,x,z\}}$ is faithful, $\chi|_{\{x,w\}}$ is faithful; cf. Lemma 2.4. Thus, as $w \in xG_1(N)$ and $z \in w\langle N\rangle$, $\chi|_{\{x\}\cup z\langle N\rangle}$ is faithful; cf. Lemma 4.1.

Now the claim follows from the fact that N has been chosen arbitrarily in $\mathcal{N}$. ∎

LEMMA 5.5: *Let x be an element in X such that $y \in S(x,z)$. Then, if $y \in xL$, each faithful map from $\{y\} \cup zV$ to X extends faithfully to $\{x,y\} \cup zV$.*

Proof: Let g stand for the uniquely determined element in G which satisfies $z \in yg$, and let l be the uniquely determined element in L which satisfies $y \in xl$. Then, as $y \in S(x,z)$, $l \in G_1(g)$.

Let χ be a faithful map from $\{y\} \cup zV$ to X, and let us denote by $\mathcal{M}$ the set of all elements N of $\mathcal{N}$ which satisfy $l \in \langle N\rangle^{g^*}$.

Assume first that $\emptyset = \mathcal{M}$. In this case, we pick an arbitrary element in $y\chi l$, and we denote this element by $x\chi$. By Lemma 4.3, the extension of χ defined in this way, must be faithful.

Let us now assume that $\emptyset \neq \mathcal{M}$, and let us denote by C the intersection of the elements in $\mathcal{M}$. Then, by Corollary 5.3, $l \in \langle C\rangle^{g^*}$, and that means that $lg \subseteq g\langle C\rangle$.

On the other hand, we have $z \in yg$ and $y \in xl$, whence $z \in xlg$. Thus, as $lg \subseteq g\langle C\rangle$, $z \in xg\langle C\rangle$. Thus, there exists an element w in xg such that $z \in w\langle C\rangle$. From $x \in yl$, $w \in xg$, and $l \in G_1(g)$ we obtain that $x \in S(y,w)$.

Let us denote by U the union of the sets $\langle M\rangle$ with $M \in \mathcal{M}$. Then, as $w \in z\langle C\rangle$, $wU = zU$. Thus, as $x \in S(y,w)$, $\chi|_{\{y\}\cup zU}$ extends faithfully to $\{x,y\}\cup zU$; cf. Lemma 5.4. Thus, by Lemma 4.3, χ is faithful also on $\{x,y\}\cup zV$. ∎

LEMMA 5.6: *Let x be an element in X. Then, each faithful map from $yV \cup \{z\}$ to X extends faithfully to $yV \cup \{x,z\}$.*

Proof: By induction, we may assume that $x \in zL$. Then the claim follows from Lemma 5.4 and Lemma 5.5. ∎

Let us denote by g the uniquely determined element in G which satisfies $z \in yg$.

For each element N in $\mathcal{N}$, let N' be a subset of L such that, for $e \in g\langle N'\rangle \cap G_1(N')$, $\langle N'\rangle \subseteq \langle N\rangle^e$. We denote by V' the union of the sets $\langle N'\rangle$ with $N \in \mathcal{N}$.

LEMMA 5.7: *Each faithful map from $yV \cup \{z\}$ to X extends faithfully to $yV \cup zV'$.*

Proof: Let χ be a faithful map from $yV \cup \{z\}$ to X. Then, by Lemma 5.6, χ extends to $yV \cup zV'$ in such a way that, for each element x in zV', $\chi|_{yV\cup\{x,z\}}$ is faithful.

Let us now fix an element N in $\mathcal{N}$, and let us pick two elements v and w in $z\langle N'\rangle$. By Lemma 5.6, $\chi|_{y\langle N\rangle\cup\{v\}}$ extends faithfully to $y\langle N\rangle \cup \{v,w\}$. By Lemma 4.4, this extension coincides with $\chi|_{y\langle N\rangle\cup\{v,w\}}$. Thus, $\chi|_{\{v,w\}}$ is faithful.

Since v and w have been chosen arbitrarily in $z\langle N'\rangle$, we have shown that $\chi|_{z\langle N'\rangle}$ is faithful. Thus, the claim follows from Lemma 4.2. ∎

THEOREM A: *Each faithful map from yV to X extends faithfully to $yV \cup zV'$.*

Proof: Let χ be a faithful map from yV to X. Then, by Lemma 5.6, χ extends faithfully to $yV \cup \{z\}$. Thus, the claim follows from Lemma 5.7. ∎

LEMMA 5.8: *Let z' be an element in $yg \cap zV'$. Then, for each faithful map χ from $yV \cup zV'$ to X, there exists a faithful map χ' from $yV \cup z'V'$ to X which coincides with χ on yV and on $zV' \cap z'V'$.*

Proof: We are assuming that $z' \in yg$. Thus, there exists a faithful map χ' from $yV \cup z'V'$ to X which coincides with χ on yV and on z'; cf. Lemma 5.7. We shall show that χ and χ' coincide on $zV' \cap z'V'$. In order to do so, we pick an element x in $zV' \cap z'V'$. Since $x \in z'V'$, there exists an element N in $\mathcal{N}$ such that $x \in z'\langle N'\rangle$.

Since $\langle N\rangle \subseteq V$, the (faithful) maps $\chi|_{y\langle N\rangle\cup\{x\}}$ and $\chi'|_{y\langle N\rangle\cup\{x\}}$ coincide on $y\langle N\rangle$. Moreover, as $x \in z'\langle N'\rangle$, $x\chi' \in z'\chi'\langle N'\rangle = z'\chi\langle N'\rangle$. Finally, as $x \in z'\langle N'\rangle$, $x\chi \in z'\chi\langle N'\rangle$. Thus, by Lemma 4.4, $x\chi = x\chi'$. ∎

6. Proof of Theorem B

It is the goal of this section to prove Theorem B. The proofs of the first two results are similar to the corresponding ones for Coxeter groups. We add them here just for the sake of completeness. Our general approach to Theorem B is inspired by the arguments of Tits' well-known reduction theorems for buildings of spherical type.

In this section, the letter N stands for a subset of L such that $\langle N \rangle$ is finite.

LEMMA 6.1: *Let g be an element in $\langle N \rangle$ satisfying $\ell(g) = \max \ell(\langle N \rangle)$. Then $g \in G_{-1}(\langle N \rangle)$.*

Proof: From $g \in \langle N \rangle$ and $\ell(g) = \max \ell(\langle N \rangle)$ we obtain that, for each element l in N, $g \notin G_1(l)$. Thus, by Lemma 1.6, $g \in G_{-1}(N)$. Thus, the claim follows from Lemma 3.4. ∎

LEMMA 6.2: *The set $\langle N \rangle$ contains a uniquely determined element g satisfying $\ell(g) = \max \ell(\langle N \rangle)$.*

Proof: Let e and f be elements in $\langle N \rangle$ satisfying $\ell(e) = \max \ell(\langle N \rangle) = \ell(f)$. We shall show that $e = f$.

Since $f \in \langle N \rangle$ and $\ell(f) = \max \ell(\langle N \rangle)$, $f \in G_{-1}(\langle N \rangle)$; cf. Lemma 6.1. Thus, as $e \in \langle N \rangle$, $f \in G_{-1}(e)$. Thus, by definition, there exists an element d in G such that $f \in de$ and $\ell(f) = \ell(d) + \ell(e)$.

Similarly, we find an element c in G such that $e \in cf$ and $\ell(e) = \ell(c) + \ell(f)$. It follows that $\ell(c) = 0$. Thus, $1 = c$, whence $e = f$. ∎

Lemma 6.2 says that the set $\langle N \rangle$ contains a uniquely determined element g satisfying $\ell(g) = \max \ell(\langle N \rangle)$. We shall denote this element by j_N.

Since $(j_N)^* \in \langle N \rangle$ and $\ell((j_N)^*) = \ell(j_N)$, we have $(j_N)^* = j_N$. In the following, we shall use this equation without further reference.

Let g be an element in $\langle N \rangle$. Then, as $j_N \in \langle N \rangle$ and $\ell(j_N) = \max \ell(\langle N \rangle)$, $j_N \in G_{-1}(g)$; cf. Lemma 6.1. Thus, by definition, G contains an element f with $j_N \in fg$ and $\ell(j_N) = \ell(f) + \ell(g)$. Moreover, by Lemma 2.1, G contains at most one such element. We denote this element by $g^{(N)}$. Note that $g^{(N)} \in \langle N \rangle$. Thus, $g^{(N)(N)}$ is defined. We write $g^{[N]}$ instead of $g^{(N)(N)}$.

It is clear that, for each element g in $\langle N \rangle$, $\ell(g^{[N]}) = \ell(g)$. In particular, for each element l in N, we have $l^{[N]} \in N$; cf. Lemma 3.3.

For each subset M of N, we define $M^{[N]}$ to be the set of all elements $l^{[N]}$ with $l \in M$.

LEMMA 6.3: *For each subset M of N, the following holds.*

(i) *We have $(j_M)^{(N)} \in j_N\langle M\rangle$.*

(ii) *We have $(j_M)^{(N)} \in G_1(M)$.*

(iii) *We have $\langle M\rangle \subseteq \langle M^{[N]}\rangle^{((j_M)^{(N)})}$.*

Proof: (i) By definition, we have $j_N \in (j_M)^{(N)} j_M$. Thus, as $j_M \in \langle M\rangle$, $j_N \in (j_M)^{(N)}\langle M\rangle$, so that $(j_M)^{(N)} \in j_N\langle M\rangle$.

(ii) In order to show that $(j_M)^{(N)} \in G_1(M)$ we pick an element l in M, and we shall show that $(j_M)^{(N)} \in G_1(l)$. Since $l \in M$, $j_M \in G_{-1}(l)$; cf. Lemma 6.1. Thus, by Lemma 1.5(iv), $G_1(j_M) \subseteq G_1(l)$. Thus, as $(j_M)^{(N)} \in G_1(j_M)$, $(j_M)^{(N)} \in G_1(l)$.

(iii) From Lemma 1.2(ii) we know that it is enough to show that, for each element l in M, $(j_M)^{(N)} l \subseteq \langle M^{[N]}\rangle (j_M)^{(N)}$.

In order to do so we pick an element in l in M. Then, by (ii), $(j_M)^{(N)} \in G_1(l)$. Thus, there exists an element e in $(j_M)^{(N)} l$ such that $\ell(e) = \ell((j_M)^{(N)}) + 1$.

Assume that $M^{[N]} \subseteq G_1(e)$. Then, by Lemma 3.5, $j_{M^{[N]}} \in G_1(e)$. Thus, there exists an element f in $j_{M^{[N]}} e$ such that $\ell(f) = \ell(j_{M^{[N]}}) + \ell(e)$. On the other hand, as $f \in \langle N\rangle$, $\ell(f) \leq \ell(j_N)$, contradiction.

Thus, as G is assumed to be a Coxeter scheme over L, there exists an element k in $M^{[N]}$ such that $(j_M)^{(N)} l = k(j_M)^{(N)} \subseteq \langle M^{[N]}\rangle (j_M)^{(N)}$. ∎

For the remainder of this section, we shall write j instead of j_N.

LEMMA 6.4: *Let (y, z) be an element in j, and let l be an element in N. Then each faithful map from $y\langle l^{[N]}\rangle \cup z\langle l\rangle$ to X is uniquely determined by its action on $y\langle l^{[N]}\rangle \cup \{z\}$.*

Proof: Let χ be a faithful map from $y\langle l^{[N]}\rangle \cup z\langle l\rangle$ to X, and let x be an element in $z\langle l\rangle$. By Lemma 6.3(ii), $l^{(N)} \in G_1(l)$, and, by Lemma 6.3(iii), $\langle l\rangle \subseteq \langle l^{[N]}\rangle^{l^{(N)}}$. Thus, the claim follows from Lemma 4.4 (applied to $\{l\}$ and $\{l^{[N]}\}$ instead of M and N). ∎

LEMMA 6.5: *Let (y, z) be an element in j, and let M be a subset of N with $\emptyset = T(M)$. Then each faithful map from $y\langle M^{[N]}\rangle \cup z\langle M\rangle$ to X is uniquely determined by its action on $yM^{[N]} \cup \{z\}$.*

Proof: Let χ be a faithful map from $y\langle M^{[N]}\rangle \cup z\langle M\rangle$ to X. From Lemma 6.4 we obtain that χ is uniquely determined on zM.

Let us now prove that χ is uniquely determined on zM^2. In order to show this we pick an element w in zM^2. Since $w \in zM^2$, there exists an element v

in zM such that $w \in vM$. Since $v \in zM$, there exists an element h in M such that $v \in zh$. Since $w \in vM$, there exists an element k in M such that $w \in vk$. We are assuming that $T(M)$ is empty. Thus, $zj \cap vj \cap y\langle h^{[N]}\rangle$ is not empty. Let x be an element in $zj \cap vj \cap y\langle h^{[N]}\rangle$.

By hypothesis, χ is uniquely defined on $yh^{[N]}$. In particular, χ is uniquely defined on x. Thus, as $x \in zj$, χ is uniquely defined on $x\langle k^{[N]}\rangle$; cf. Lemma 6.4. Thus, as $v \in xj$, χ is uniquely defined on $v\langle k\rangle$; cf. Lemma 6.4 once again.

Induction now finishes our proof of the lemma. ∎

For the remainder of this section, we fix an element n in $\{1, \ldots, |N|\}$. The letter $\mathcal{M}$ will stand for the set of all subsets of N of order less or equal to n. By V we denote the union of the sets $\langle M\rangle$ with $M \in \mathcal{M}$.

For each pair (y, z) in j, we define X_{yz} to be the set of all faithful maps from $yV \cup zV$ to X.

From now on, we assume $T(N)$ to be empty. As a consequence of Lemma 6.5 we obtain the following.

Corollary 6.6: *Let (y, z) be an element in j. Then each map in X_{yz} is uniquely determined by its action on $yN \cup \{z\}$.*

Let M be an element in $\mathcal{M}$, let (v, w) and (y, z) be elements in j such that $y \in v\langle M^{[N]}\rangle$ and $z \in w\langle M\rangle$.

Two maps χ in X_{vw} and ψ in X_{yz} will be called M-compatible if, for each element x in $wj \cap zj \cap v\langle M^{[N]}\rangle$, there exist maps η in X_{xw} and ζ in X_{xz} such that χ and η coincide on $vV \cap xV$ and on wV, η and ζ coincide on xV and on $wV \cap zV$, and ζ and ψ coincide on $xV \cap yV$ and on zV.

Note that, for any two elements y and z in X, $\emptyset \neq yj \cap zj$. (This follows by induction.) Thus, if (v, w) and (y, z) are elements in j, ϕ an element in X_{vw}, and ψ an element in X_{yz} which is M-compatible with ϕ, $z\psi \in w\phi\langle M\rangle$ and $y\psi \in v\phi\langle M^{[N]}\rangle$.

We define X to be the union of the sets X_{yz} with $(y, z) \in j$.

For the remainder of this section, we assume that $2 \leq n$.

Lemma 6.7: *M-compatibility is an equivalence relation on* X.

Proof: By Lemma 5.8, M-compatibility is reflexive. That M-compatibility is symmetric follows immediately from the definition of M-compatibility. Let us prove transitivity.

Let (u', u), (v', v), (w', w) be elements in j such that v, $w \in u\langle M\rangle$ and v', $w' \in u'\langle M^{[N]}\rangle$. Let χ_u be an element in $\mathrm{X}_{u'u}$, let χ_w be an element in $\mathrm{X}_{w'w}$, and let χ_v be an element in $X_{v'v}$ which is M-compatible with both, χ_u and χ_w. We have to show that χ_u and χ_w are M-compatible. By induction, we may assume that $w \in vM$. Thus, there exists an element l in M such that $w \in vl$.

In order to show that χ_u and χ_w are M-compatible, we pick an element t in $uj \cap wj \cap u'\langle M^{[N]}\rangle$. Then, as we are assuming that $T(N)$ is empty, there exist elements r in $uj \cap vj \cap t\langle l^{[N]}\rangle$ and s in $vj \cap wj \cap t\langle l^{[N]}\rangle$; recall that $w \in vl$.

Since χ_u and χ_v are assumed to be compatible, there exist maps ρ from $rV \cup uV$ to X and σ from $rV \cup vV$ to X such that χ_u and ρ coincide on $u'V \cap rV$ and on uV, ρ and σ coincide on rV and on $uV \cap vV$, and σ and χ_v coincide on $rV \cap v'V$ and on vV.

Similarly, as χ_v and χ_w are assumed to be compatible, there exist maps ζ from $sV \cup vV$ to X and η from $sV \cup wV$ to X such that χ_v and ζ coincide on $v'V \cap sV$ and on vV, ζ and η coincide on sV and on $vV \cap wV$, and η and χ_w coincide on $sV \cap w'V$ and on wV.

By Lemma 5.8, there exists a faithful map ϕ from $tV \cup uV$ to X which coincides with χ_u on $u'V \cap tV$ and on uV. Similarly, we find a faithful map ψ from $tV \cup wV$ to X which coincides with ϕ on tV and on $uV \cap wV$. Finally, referring to Lemma 5.8 a third time, we obtain a faithful map $\tilde{\eta}$ from $sV \cup wV$ to X which coincides with ψ on $tV \cap sV$ and on wV.

We claim that $\tilde{\eta} = \eta$, and in order to see this we pick an element x in sN. Then, as we are assuming that $2 \leq n$,

$$x\tilde{\eta} = x\psi = x\phi = x\rho = x\sigma = x\zeta = x\eta.$$

Since x has been chosen arbitrarily in sN, we have that $\tilde{\eta}|_{sN} = \eta|_{sN}$. Thus, as

$$w\tilde{\eta} = w\psi = w\phi = w\rho = w\sigma = w\zeta = w\eta,$$

$\tilde{\eta} = \eta$; cf. Corollary 6.6.

The maps ψ and $\tilde{\eta}$ coincide on w' and on wV. The same is true for η and χ_w. Thus, ψ and χ_w coincide on w' and on wV. Thus, they coincide on $tV \cap w'V$ and on wV. ∎

Let (y, z) be an element in j, and let M be an element in $\mathcal{M}$. From Lemma 6.7 we obtain (in particular) that each M-equivalence class intersects X_{yz} in at most one element. The following lemma shows that, if $\emptyset \neq X$, each M-equivalence class intersects X_{yz} in at least one element.

LEMMA 6.8: *Let (y, z) and (y', z') be elements in j, and let M be an element in $\mathcal{M}$ such that $y' \in y\langle M^{[N]}\rangle$ and $z' \in z\langle M\rangle$. Then, for each element χ in X_{yz}, there exists an element in $X_{y'z'}$ which is M-compatible with χ.*

Proof: By Lemma 6.7, we may assume that $z' \in zM$. In this case, there exists an element l in M such that $z' \in zl$.

Since $z' \in zl$, we may assume that $y' \in yl^{[N]}$. Since we are assuming that $T(N)$ is empty, there exists an element x in $zj \cap z'j \cap y\langle l^{[N]}\rangle$.

By Lemma 5.8, there exists a faithful map ϕ from $xV \cup zV$ to X which coincides with χ on $yV \cap xV$ and on vV. Similarly, we find a faithful map ψ from $xV \cup z'V$ to X which coincides with ϕ on xV and on $zV \cap z'V$. Finally, referring to Lemma 5.8 a third time, we obtain a faithful map $\chi_{z'}$ from $y'V \cup z'V$ to X which coincides with ψ on $xV \cap y'V$ and on $z'V$.

The maps χ and ϕ coincide on $yV \cap xV$, ϕ and ψ coincide on xV, and ψ and $\chi_{z'}$ coincide on $xV \cap y'V$. On the other hand, we know from Lemma 3.11 that $yV \cap y'V \subseteq xV$. Thus, χ and $\chi_{z'}$ coincide on $yV \cap y'V$.

The maps χ and ϕ coincide on zV, ϕ and ψ coincide on $zV \cap z'V$, and ψ and $\chi_{z'}$ coincide on $z'V$. Thus, χ and $\chi_{z'}$ coincide on $zV \cap z'V$.

In order to show that χ and $\chi_{z'}$ are M-compatible, we pick an element t in $y'\langle M^{[N]}\rangle \cap zj \cap z'j$. By Lemma 5.8, there exists a faithful map ϕ from $tV \cup zV$ to X such that χ and ϕ coincide on $yV \cap tV$ and on zV. Similarly, there exists a faithful map ψ from $tV \cup z'V$ to X such that ϕ and ψ coincide on tV and on $zV \cap z'V$.

We claim that ψ and χ_w coincide on $z'N$. In order to see this we pick an element x in $z'N$. Then, as $z' \in zM$, $x\psi = x\phi = x\chi = x\chi_{z'}$. Thus, as $y'\psi = y'\phi = y'\chi = y'\chi_w$, we conclude that ψ and $\chi_{z'}$ coincide on $tV \cap y'V$ and on $z'V$. ∎

We call N-compatibility the smallest equivalence relation on X which contains M-compatibility for each M in $\mathcal{M}$.

Let (y, z) and (y', z') be elements in j, let ϕ be an element in X_{yz}, and let ψ be an element in $\mathrm{X}_{y'z'}$. Then, if ϕ and ψ are N-compatible, $z' \in z\langle N\rangle$ (and $y' \in y\langle N\rangle$).

Lemma 6.9: *Let (y, z) be an element in j. Then, for each element χ in X there exists at most one element in X_{yz} which is N-compatible with χ.*

Proof: Let us denote by F the set of all elements g in $\langle N\rangle$ such that zg contains an element contradicting our claim. By way of contradiction, we assume that $\emptyset \neq F$. We pick an element f in F such that $\min \ell(F) = \ell(f)$.

Since $f \in F$, $1 \neq f$. Thus, by Lemma 1.3, there exist elements e in $\langle N\rangle$ and h in N such that $f \in eh$ and $\ell(f) = \ell(e) + 1$.

Since $f \in F$, there exists an element x in zf such that X_x contains at least two (different) elements χ_x and χ'_x which are N-compatible with χ.

From $x \in zf$ and $f \in eh$ we obtain that $x \in zeh$. Thus, there exists an element u in ze such that $x \in uh$.

By Lemma 6.8, there exists an element χ_u in X_u which is N-compatible with χ_x. Similarly, we find an element χ'_u in X_u which is N-compatible with χ'_x. Both, χ_u and χ'_u are N-compatible with χ. Thus, as $\min \ell(F) = \ell(f)$, $\chi_u = \chi'_u$. Thus, by Lemma 6.7, $\chi_x = \chi'_x$. ∎

THEOREM B: *Let x be an element in X. Then each faithful map from xV to X extends faithfully to $x\langle N\rangle$.*

Proof: Let χ be a faithful map from xV to X, and let x' be an element in $x'j$. Then, by Theorem A, χ extends faithfully to a map χ_x from $x'V \cup xV$ to X. Thus, by Lemma 6.9, there exists, for each element w in $x\langle N\rangle$ a uniquely determined element χ_w in X_w which is N-compatible with χ_x.

For each element w in $x\langle N\rangle$, we set $w\chi := w\chi_w$. Then we obtain that, for any two elements u and v in $x\langle N\rangle$ satisfying $v \in uN$, $v\chi \in u\chi N$. Thus, χ is a faithful map from $x\langle N\rangle$ to X.

Let t be an element in xV. Then, by definition, there exists an element M in $\mathcal{M}$ such that $t \in x\langle M\rangle$. Thus, by induction, $\chi_t|_{x\langle M\rangle} = \chi|_{x\langle M\rangle}$. This shows that χ extends χ_x.

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